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On Uniqueness of Best Spline Approximations with Free Knots

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This paper is concerned with Chebyshev approximation by spline functions with free knots. If a zero of a Chebyshev spline function occurs at a knot, the multiplicity of the zero is suitably extended. Theorems on uniqueness on the whole approximation interval and on subintervals are stated in terms of alternation properties.

1. INTRODUCTION

In this paper the approximation to a real function $f \in C[a, b]$ by Chebyshev spline functions is considered. Spline functions are defined as follows (cf. [4]): Given $n + 1$ positive functions $w_i \in C^{n-i}[a, b]$, $i = 0, 1,..., n$, let

$$\phi_l(t, x) = \begin{cases} w_0(t) \int_x^t w_1(\xi_1) \int_x^{\xi_1} w_2(\xi_2) \cdots \int_x^{\xi_{l-1}} w_l(\xi_l)\, d\xi_l \cdots d\xi_1, & t \geqslant x, \\ 0, & t < x, \end{cases}$$

$$u_l(t) = \phi_l(t, a), \qquad l = 0, 1,..., n.$$

Then,

$$S_{n,k} = \left\{ s(t) \,\middle|\, s(t) = \sum_{i=0}^{n} a_i u_i(t) + \sum_{i=1}^{r} \sum_{j=1}^{m_i} b_{ij} \phi_{n-j+1}(t, y_i), \right.$$
$$\left. a = y_0 < y_1 < \cdots < y_{r+1} = b,\ 1 \leqslant m_i \leqslant n + 1,\ \sum_{i=1}^{r} m_i \leqslant k \right\}$$

is the class $S_{n,k}$ of Chebyshev spline functions of order n with the parameters a_k, b_{ij}, m_i, y_i ($k = 0,..., n$; $i = 1,..., r$; $j = 1,..., m_i$). In the case $w_0(t) \equiv 1$, $w_i(t) \equiv i$, $i = 1,..., n$, $S_{n,k}$ reduces to the class of polynomial spline functions with

$$\phi_l(t, x) = (t - x)_+^l .$$

According to Schumaker [4], there always exists a best approximation $s^* \in S_{n,k}$ to $f \in C[a, b]$, i.e.,

$$\|f - s^*\| \leqslant \|f - s\| = \sup\{|(f - s)(t)| \mid t \in [a, b]\}$$

holds for every $s \in S_{n,k}$. At least one best approximation is continuous.

It is our aim to establish sufficient conditions which guarantee that best approximation is unique on the whole approximation interval or on a subinterval. As usual, these conditions will involve alternation properties.

2. Preliminaries

Interpolation with Chebyshev spline functions (with fixed knots) leads to a linear system of equations, the determinant of which has been studied by Karlin and Ziegler [2].

Let $T = \{t_1 \leqslant t_2 \leqslant \cdots \leqslant t_m\}$ and $X = \{x_1 \leqslant x_2 \leqslant \cdots \leqslant x_m\}$ such that

(1) No more than $n + 1$ elements of T (or X) coincide.

(2) If i elements of T coincide with j elements of X, then $i + j \leqslant n + 2$.

Define

$$\phi(T, X) = \phi\begin{pmatrix} t_1, \ldots, t_m \\ x_1, \ldots, x_m \end{pmatrix} = \det(\phi_n(t_i, x_j)_{i,j=1}^m) \tag{2.1}$$

with the following interpretation:

(a) If $x_{i-1} < x_i = x_{i+1} = \cdots = x_{i+p} < x_{i+p+1}$, then the $(i + j)$th column vector has to be replaced by $[\phi_{n+j-p}(t_1, x_i), \ldots, \phi_{n+j-p}(t_m, x_i)]^T$ for $j = 1, \ldots, p$.

(b) If coincidences of elements of T occur, successive rows of (2.1) are replaced by derivatives of the previous rows.

With these conventions, the following lemma has been shown by Karlin and Ziegler [2]:

Lemma 2.1. *The determinant* (2.1) *is nonnegative and is strictly positive if and only if*

$$t_{i-n-1} < x_i < t_i, \qquad i = 1, 2, \ldots, m, \tag{2.2}$$

where the left-hand inequality is ignored for $i \leqslant n + 1$; *in the case* $n = 0$, *equality is permitted on the right-hand side of* (2.2).

With Lemma 2.1, the zero structure of Chebyshev spline functions can be studied, paying attention to the fact that spline functions may vanish identically on subintervals. A zero z of $s \in S_{n,k}$ may be counted p times ($p \leqslant n$) if the first $p - 1$ derivatives vanish. If, in addition, z coincides with a knot of multiplicity $n - p + 1$, then the zero may be counted even $p + 1$ times. (Special cases are considered in [1] and [3].)

By $B_{n,k}$ we denote the minimal deviation $\| f - s^* \|$. The notation $s \in S_{n,k}(x_1, ..., x_k)$ indicates that s has the knots $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_k$ repeated according to their multiplicity.

LEMMA 2.2. *Let $s \in S_{n,k}(x_1, ..., x_k) \cap C[a, b]$. If s possesses $n + k + 1$ zeros $z_1 \leqslant z_2 \leqslant \cdots \leqslant z_{n+k+1}$ on $[a, b]$ satisfying*

$$z_i < x_i < z_{n+i+1}, \qquad i = 1, ..., k, \tag{2.3}$$

then s vanishes identically on $[a, b]$.

Proof. If there are no zeros of multiplicity $p + 1$ at knots of multiplicity $n - p + 1$, the result follows immediately from Lemma 2.1. If there are such zeros, let x_q be the left most and assume

$$x_{q-1} < x_q = x_{q+1} = \cdots = x_{q+n-p} < x_{q+n-p+1} .$$

In view of (2.3) for $i = q$ and $i = q + n - p$, and considering the restrictions of s on $[a, x_q]$ and $[x_q, b]$, the proof is easily done by induction. ∎

The following lemma reduces to Lemma 2.2 in [4] provided that there are only simple zeros. If zeros at knots of multiplicity n are counted twice, the second statement is due to Braess [1].

LEMMA 2.3. *Let $s \in S_{n,k}(x_1, ..., x_k) \cap C[a, b]$.*

(1) *If s possesses $n + k$ zeros $z_1 \leqslant z_2 \leqslant \cdots \leqslant z_{n+k}$ and does not vanish identically between two of them, then*

$$z_i < x_i < z_{n+i}, \qquad i = 1, ..., k. \tag{2.4}$$

(2) *If s possesses $n + k + 1$ zeros $z_1 \leqslant z_2 \leqslant \cdots \leqslant z_{n+k+1}$, then s vanishes identically between two of them.*

Proof. The proof proceeds by induction on k. For $k = 0$ the result follows from Lemma 2.2. Assume the statements proved for $0, 1, ..., k - 1$. If the right-hand side of (2.4) does not hold, then, in view of $x_q \geqslant z_{n+q}$ for some q, there are $n + q$ zeros of $s \in S_{n,q-1}[a, x_q]$, contrary to the induction hypothesis. The other case is argued similarly.

If s possesses $n + k + 1$ zeros but does not vanish identically between two of them, then (1) implies $z_i < x_i < z_{n+i}$ and therefore $z_i < x_i < z_{n+i+1}$

for $i = 1,..., k$. By virtue of Lemma 2.2 we have $s \equiv 0$ on $[a, b]$, a contradiction to the assumptions of (2). ▌

A function $f \in C[a, b]$ is said to alternate m times on $[a, b]$ if there exist $m + 1$ points $a \leqslant t_1 < t_2 < \cdots < t_{m+1} \leqslant b$ with

$$|f(t_i)| = \|f\|, \quad i = 1,..., m + 1, \quad f(t_i) = -f(t_{i+1}), \quad i = 1,..., m.$$

An immediate consequence of Theorem 2.2 of Braess [1] is the following theorem, which shows that under certain conditions adding further knots does not lead to a better approximation.

THEOREM 2.4. *Let $f \in C[a, b]$ and $s \in S_{n,k} \cap C[a, b]$ have knots $a = y_0 < y_1 < \cdots < y_{r+1} = b$. If $f - s$ alternates $n + k + l + m + 1$ times on some subinterval $[y_p, y_q]$ where $s \in S_{n,l}[y_p, y_q]$ holds, then s is a best approximation in $S_{n,k+m}$ and*

$$B_{n,k} = B_{n,k+1} = \cdots = B_{n,k+m}.$$

3. UNIQUENESS

One necessary condition for uniqueness of the best approximation, being $B_{n,k} < B_{n,k-1}$, has been developed by Schumaker [4]. A weaker condition is that the best approximation $s \in S_{n,k}$ is not contained in $S_{n,k-1}$. However, both conditions are not sufficient as simple examples show.

The following lemma serves for the proof of uniqueness on the whole interval while the second lemma prepares a theorem on uniqueness on a subinterval.

LEMMA 3.1. *Let*

$$s \in S_{n,k}(x_1,..., x_k) \cap C[a, b], \quad s \notin S_{n,k-1}, \quad \text{and} \quad s^* \in S_{n,k} \cap C[a, b].$$

If $\Delta = s - s^*$ possesses $n + 2k + 1$ zeros $z_1 \leqslant z_2 \leqslant \cdots \leqslant z_{n+2k+1}$ with

$$z_{2i} < x_i < z_{n+2i}, \quad i = 1,..., k, \tag{3.1}$$

then Δ vanishes identically on $[a, b]$.

Proof. The proof proceeds by induction on k. For $k = 0$ the result follows from Lemma 2.2. Assume the result proved for $k = 0, 1,..., K - 1$. We show it for $k = K$.

We can assume $\Delta \in S_{n,2k}(y_1,..., y_{2k})$ with $\{x_1,..., x_k\} \subset \{y_1,..., y_{2k}\}$ (if necessary we add virtual knots). Since Δ has $n + 2k + 1$ zeros, Lemma 2.3 applies to assert $\Delta \equiv 0$ on some subinterval $[y_p, y_q]$ of $[a, b]$.

Case 1. At first we consider the case $x_1 < x_k$ and $[y_p, y_q] \subset [x_1, x_k]$. Let $x^* \in (y_p, y_q)$, but $x^* \notin \{y_1, ..., y_{2k}\}$. Then $s \in S_{n,k_1}[a, x^*]$, $s \in S_{n,k_2}[x^*, b]$, $s^* \in S_{n,l_1}[a, x^*]$, and $s^* \in S_{n,l_2}[x^*, b]$. The choice of x^* leads to $k_1 < k$ and $k_2 < k$. Without loss of generality we can assume $l_1 \leqslant k_1$. With the zeros $z_1, z_2, ..., z_{2k_1}$ and $n+1$ zeros in $[x_{k_1}, x^*]$, the induction hypothesis applied to the interval $[a, x^*]$ yields $\Delta \equiv 0$ on $[a, x^*]$. Moreover, we have $l_1 = k_1$. Hence, $s \notin S_{n,k-1}$ implies $l_2 \leqslant k_2$, and we conclude that $\Delta \equiv 0$ on $[x^*, b]$.

Case 2. Let $x_1 < x_k$ and assume that Δ does not vanish identically on some subinterval of $[x_1, x_k]$. Then Δ vanishes identically on $[y_p, y_q] \subset [a, x_1]$ and/or on $[y_r, y_s] \subset [x_k, b]$, but does not vanish identically on some subinterval of $[y_q, y_r]$. Let m_q and m_r be the multiplicities of y_q and y_r, respectively. Then $\Delta \in S_{n,2k-m_q-m_r}[y_q, y_r]$ possesses a zero of multiplicity $n+1-m_i$ in y_i, $i = q, r$. By virtue of (3.1) there are at least $2k-(n+1)$ zeros of Δ in (x_1, x_k) (the case $2k < n+1$ is similar). Hence, Δ possesses at least

$$n+1-m_q+2k-(n+1)+n+1-m_r = n+2k-m_q-m_r+1$$

zeros on $[y_q, y_r]$. By Lemma 2.3, Δ vanishes identically on some subinterval of $[y_q, y_r]$, contrary to our assumptions. Therefore Case 1 is valid.

Case 3. Let $x_1 = x_k$. If Δ vanishes identically in an open neighbourhood of x_1, then $s \notin S_{n,k-1}$ implies that s^* has a knot of multiplicity k at $x_1 = x_k$, too. Therefore we have $\Delta \equiv 0$ on $[a, b]$.

Assume that Δ does not vanish identically in some open neighbourhood of x_1. Then the arguments of Case 2 lead to a contradiction. ∎

Condition (3.1) cannot be weakened to

$$z_{2i-1} < x_i < z_{n+2i+1}, \qquad i = 1, ..., k, \tag{3.1*}$$

as the following example shows: Let $n \geqslant 1$, $s(t) = \phi_n(t, x_1)$, $s^*(t) = c\phi_n(t, y_1)$ with $x_1 < y_1$. Define $z_1 < \cdots < z_{n+2}$ with $z_i \in [a, x_1]$, $i = 1, ..., n+2$. Choose c sufficiently great such that $\Delta = s - s^*$ has a zero z_{n+3} in (y_1, b) [in the case of polynomial splines it suffices to choose $c \geqslant (b-x_1)^n/(b-y_1)^n$]. Then Δ satisfies (3.1*) but does not vanish identically on $[a, b]$.

The following lemma can be shown by applying the same technique used in the proof of the previous lemma.

LEMMA 3.2. *Let*

$$s \in S_{n,k}(x_1, ..., x_k) \cap C[a, b], \qquad s \notin S_{n,k-1}, \qquad \textit{and} \qquad s^* \in S_{n,l} \cap C[a, b]$$

with $l \geqslant k$. *If* $\Delta = s - s^*$ *possesses* $n + k + l + 1$ *zeros* $z_1 \leqslant z_2 \leqslant \cdots \leqslant z_{n+k+l+1}$ *with*

$$z_{l-k+2i} < x_i < z_{n+2i}, \qquad i = 1,..., k, \tag{3.2}$$

then there exists a $\delta > 0$ *such that* Δ *vanishes identically on* $[x_1 - \delta, x_k + \delta]$.

Now a statement on uniqueness on a subinterval is established.

THEOREM 3.3. *Let* $f \in C[a, b]$ *and* $s \in S_{n,k}(x_1,..., x_k) \cap C[a, b]$ *with* $s \in S_{n,l}[x_p, x_q]$ *for some subinterval* $[x_p, x_q]$. *Suppose that* $s \notin S_{n,l-1}[x_p, x_q]$ *and* $f - s$ *alternates* $n + k + l + 1$ *times on* $[x_p, x_q]$ *but does not alternate* $n + 2i + 1$ *times on any subinterval of* $[x_p, x_q]$ *containing less than* $i + 1$ *knots in its interior,* $0 \leqslant i < l$. *If* s^* *is a best approximation to* f *in* $S_{n,k} \cap C[a, b]$, *then* s *and* s^* *coincide in an open neighbourhood of* $[x_{p+1}, x_{q-1}]$, *and* s *is at least* r *times differentiable on* $[x_{p+1}, x_{q-1}]$, $r \geqslant (n + k - l)/2$.

Proof. Since $f - s$ alternates $n + k + l + 1$ times on $[x_p, x_q]$ and $s \in S_{n,l}[x_p, x_q]$ holds, Theorem 2.4 yields that s is a best approximation to f in $S_{n,k}$.

Let $t_1 < t_2 < \cdots < t_{n+k+l+2}$ be the points of alternation of $f - s$ on $[x_p, x_q]$. The assumptions concerning alternation on subintervals imply

$$t_{k-l+2i+1} < x_{p+i} < t_{n+2i}, \qquad i = 1,..., l. \tag{3.3}$$

Since $\Delta = s - s^*$ is contained in $S_{n,k+l}[x_p, x_q]$ and $f - s$ alternates $n + k + l + 1$ times on $[x_p, x_q]$, there exist at least $n + k + l + 1$ zeros $z_1 \leqslant z_2 \leqslant \cdots \leqslant z_{n+k+l+1}$ of Δ on $[x_p, x_q]$ satisfying

$$z_{k-l+2i} < x_{p+i} < z_{n+2i}, \qquad i = 1,..., l,$$

(where the zeros can be chosen as counted at most twice). In view of $s \notin S_{n,l-1}[x_p, x_q]$, Lemma 3.2 implies the existence of a $\delta > 0$ such that Δ vanishes identically on $[x_{p+1} - \delta, x_{q-1} + \delta]$.

If the multiplicities of the zeros of s in $[x_p, x_q]$ are at most r, then s is $n - r$ times differentiable on $[x_p, x_q]$. Let

$$x_p \leqslant x_{p+i_0} < x_{p+i_0+1} = \cdots = x_{p+i_0+r} < x_{p+i_0+r+1} \leqslant x_q.$$

Inserting $i = i_0 + 1$ and $i = i_0 + r$ in (3.3) implies

$$t_{k-l+2(i_0+r)+1} < t_{n+2(i_0+1)},$$

and

$$r < \frac{n + 1 + l - k}{2}.$$

Hence, s is $n - r$ times differentiable on $[x_q, x_p]$ with $n - r \geqslant (n + k - l)/2$. ∎

In the particular case, when $l = k$, we obtain uniqueness on the whole approximation interval.

COROLLARY 3.4. *Let $f \in C[a, b]$ and $s \in S_{n,k} \cap C[a, b]$, but $s \notin S_{n,k-1}$. Suppose that $f - s$ alternates $n + 2k + 1$ times on $[a, b]$ but does not alternate $n + 2i + 1$ times on any subinterval containing less than $i + 1$ knots in its interior with $0 \leqslant i < k$. Then s is the unique best approximation to f in $S_{n,k}$ and s is r times differentiable with $r \geqslant n/2$.*

The above Corollary is similar to a theorem of Schumaker. His proof makes use of Lemma 5.2 in [4], which contradicts the following example: Let $[a, b] = [0, 3]$, $y_1 = 1$, $y_2 = 2$, and define

$$s(t) = a_1\phi_n(t, 1) + b_1\phi_n(t, 2) \in S_{n,2}, \qquad a_1 \neq 0, \quad b_1 \neq 0,$$
$$s^*(t) = s(t) + c_1\phi_n(t, 2) \in S_{n,2}, \qquad c_1 \neq 0.$$

If $n = k = 2$, then $\Delta = s - s^* \in S_{n,k+0}$ possesses $n + k + 0 + 1$ zeros,

$$0 < z_1 < z_2 < z_3 < 1 < z_4 < z_5 < 2,$$

which satisfy the assumptions of the lemma, but Δ does not vanish identically on [0, 3].

A simple example for uniqueness on a subinterval follows.

EXAMPLE 3.5. Let $f \in C[-2, 1]$ be the polygon connecting the points

$(-2; 0)$, $(-1; 1)$, $(-3/4; -1)$, $(-1/2; 1)$, $(-1/4; -1)$,
$(1/4; 65/64)$, $(1/2; -7/8)$, $(3/4; 91/64)$, $(1; 0)$
(see Figure 1),

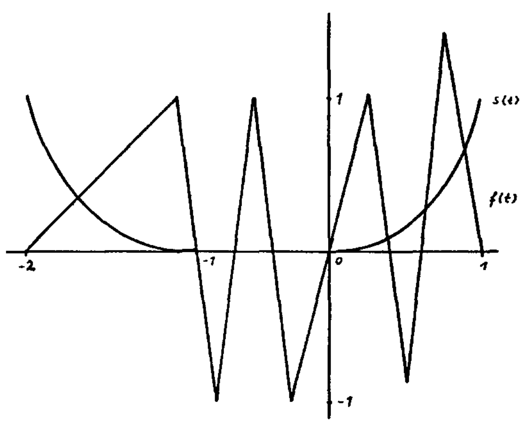

FIGURE 1

and consider the approximation to f in $S_{3,2}$. Define

$$s(t) = -(t+1)^3 + (t+1)_+^3 + (t-0)_+^3 \in S_{3,2}.$$

$f - s$ alternates $n + k + l + 1$ times on $[-1, 1]$ (with $n = 3$, $k = 2$, $l = 1$) and n times on $[-1, 0]$ and on $[0, 1]$. Hence, by Theorem 2.4, the spline s is a best approximation to f in $S_{n,k}$. By Theorem 3.3 the knot 0 is uniquely determined and s is contained in $C^2[-1, 1]$. Obviously we have no uniqueness on the whole interval.

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